

A lifting of an automorphism of a K3 surface over odd characteristic

Junmyeong Jang

Abstract

In this paper, we prove that, over an algebraically closed field of odd characteristic, a weakly tame automorphism of a K3 surface of finite height can be lifted over the ring of Witt vectors of the base field. Also we prove that a non-symplectic tame automorphism of a supersingular K3 surface or a symplectic tame automorphism of a supersingular K3 surface of Artin-invariant at least 2 can be lifted over the ring of Witt vectors. Using these results, we prove, for a weakly tame K3 surface of finite height, there is a lifting over the ring of Witt vectors to which whole the automorphism group of the K3 surface can be lifted. Also we prove a K3 surface equipped with a purely non-symplectic automorphism of a certain order is unique up to isomorphism.

1 Introduction

For an algebraic complex K3 surface X , the second integral singular cohomology $H^2(X, \mathbb{Z})$ is a free abelian group of rank 22 equipped with a lattice structure induced by the cup product. As a lattice

$$H^2(X, \mathbb{Z}) = U^3 \oplus E_8,$$

here U is a unimodular hyperbolic lattice of rank 2 and E_8 is a negative definite unimodular root lattice of rank 8. By the Lefschetz (1,1) theorem, the Neron-Severi group of X , $NS(X)$ is a primitive sublattice of $H^2(X, \mathbb{Z})$ and

$$NS(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$$

in $H^2(X, \mathbb{C})$. In particular the rank of $NS(X)$ is at most 20. We say the rank of $NS(X)$ is the Picard number of X and it is denoted by $\rho(X)$. $NS(X)$ is an even integral lattice of signature $(1, \rho(X) - 1)$. We say the orthogonal complement of the embedding

$$NS(X) \hookrightarrow H^2(X, \mathbb{Z})$$

the transcendental lattice of X and we denote the transcendental lattice of X by $T(X)$. $T(X)$ is an integral lattice of signature $(2, 20 - \rho(X))$. By the Hodge decomposition, $H^0(X, \Omega_{X/\mathbb{C}}^2)$ is a direct factor of $T(X) \otimes \mathbb{C}$ and there exists a projection

$$T(X) \otimes \mathbb{C} \rightarrow H^0(X, \Omega_{X/\mathbb{C}}^2).$$

By the Torelli theorem for complex K3 surfaces, an isometry $\psi \in O(H^2(X, \mathbb{Z}))$ is induced by an automorphism of X if and only if ψ preserves the line of holomorphic 2 forms $H^0(X, \Omega_{X/\mathbb{C}}^2)$ in $H^2(X, \mathbb{Z}) \otimes \mathbb{C}$ and the ample cone inside $NS(X) \otimes \mathbb{R}$. Let

$$\chi_X : \text{Aut}(X) \rightarrow O(T(X))$$

and

$$\rho_X : \text{Aut}(X) \rightarrow Gl(H^0(X, \Omega_{X/\mathbb{C}}^2))$$

be the representations of the automorphism group of X on the transcendental lattice and the global two forms respectively. Since $H^0(X, \Omega_{X/\mathbb{C}}^2)$ is a direct factor of $T(X) \otimes \mathbb{C}$, there is a canonical projection

$$p_X : \text{Im } \chi_X \rightarrow \text{Im } \rho_X.$$

It is known that p_X is isomorphic and $\text{Im } \chi_X$ and $\text{Im } \rho_X$ are finite cyclic groups ([21]). Assume the order of $\text{Im } \rho_X$ is N and $\xi_N = \rho_X(\alpha)$ is a primitive N -th root of unity. Then in a natural way, $T(X)$ is a free $\mathbb{Z}[\xi_N]$ -module and the rank of $T(X)$ is a multiple of $\phi(N)$ ([18]). Here ϕ is the Euler ϕ -function.

An automorphism $\alpha \in \text{Aut}(X)$ is symplectic if $\rho_X(\alpha) = 1$. An automorphism α is purely non-symplectic if α is of finite order greater than 1 and the order of α is equal to the order of $\rho_X(\alpha)$.

Assume k is an algebraically closed field of odd characteristic p . Let W be the ring of Witt vectors of k and K be the fraction field of W . Assume X is a K3 surface over k . The formal Brauer group of X , \widehat{Br}_X is a smooth one dimensional formal group over k and the height of \widehat{Br}_X is an integer between 1 and 10 or ∞ .

If the height of X is ∞ , we say X is supersingular and it is known that $\rho(X) = 22$ ([3], [19], [20]). The discriminant group of $NS(X)$, $(NS(X))^*/NS(X)$ is $(\mathbb{Z}/p)^{2\sigma}$ for an integer σ between 1 and 10. We call σ the Artin-invariant of X . It is known that the lattice structure of $NS(X)$ is determined by the base characteristic p and the Artin-invariant ([25]). All the supersingular K3 surfaces of Artin-invariant σ form a family of $\sigma - 1$ dimension over k and a supersingular K3 surface of Artin-invariant 1 is unique up to isomorphism [24].

If X is of finite height h , the second crystalline cohomology has a slope decomposition ([9], [10], [14])

$$H_{cris}^2(X/W) = H_{cris}^2(X/W)_{[1-1/h]} \oplus H_{cris}^2(X/W)_{[1]} \oplus H_{cris}^2(X/W)_{[1+1/h]}.$$

Considering the slope spectral sequence, $H_{cris}^2(X/W)_{[1-1/h]}$ is $H^2(X, W\mathcal{O}_X)$ which is isomorphic to the Dieudonné module of \widehat{Br}_X . The Dieudonné module of a 1 dimensional smooth formal group of finite height h can be express as

$$W[V, F]/(VF = p, F = V^{h-1}).$$

Here F is a Frobenius linear operator and V is a Frobenius inverse linear operator. It follows that $H_{cris}^2(X/W)_{[1-1/h]}$ is a free W -module of rank h . For the cup product pairing, $H_{cris}^2(X/W)_{[1-1/h]}$ and $H_{cris}^2(X/W)_{[1+1/h]}$ are dual to each other and $H_{cris}^2(X/W)_{[1]}$ is unimodular. Therefore the rank of $H_{cris}^2(X/W)_{[1]}$ is $22 - 2h$. Considering the cycle map

$$c : NS(X) \otimes W \hookrightarrow H_{cris}^2(X/W)_{[1]}.$$

we have $\rho(X) \leq 22 - 2h$. We call the orthogonal complement of the embedding

$$c : NS(X) \otimes W \hookrightarrow H_{cris}^2(X/W)$$

the crystalline transcendental lattice of X and it is denoted by $T_{cris}(X)$. Since

$$H_{cris}^2(X/W)_{[1-1/h]} \oplus H_{cris}^2(X/W)_{[1+1/h]}$$

is a direct factor of $T_{cris}(X)$ and there is an isomorphism $H^2(X, \mathcal{O}_X)/V \simeq H^2(X, \mathcal{O}_X)$, we have a canonical projection $T_{cris}(X) \rightarrow H^2(X, \mathcal{O}_X)$. We denote the representation of $\text{Aut}(X)$ on T_{cris} by

$$\chi_{cris, X} : \text{Aut}(X) \rightarrow O(T_{cris}(X)).$$

By the Serre duality, the representation of $\text{Aut}(X)$ on $H^2(X, \mathcal{O}_X)$ is isomorphic to ρ_X and there is a compatible projection

$$p_{cris, X} : \text{Im } \chi_{cris, X} \rightarrow \text{Im } \rho_X.$$

For any $\alpha \in \text{Aut}(X)$, the characteristic polynomial of $\alpha^*|H_{cris}^2(X/W)$ has integer coefficients ([8], 3.7.3). Hence the characteristic polynomial of $\chi_{cris, X}(\alpha)$ also has integer coefficients.

For a K3 surface X of finite height over k , there is a Neron-Severi group preserving lifting \mathfrak{X}/W ([22], [17], [11]). When $X_{\bar{K}} = \mathfrak{X} \otimes \bar{K}$ is a geometric generic fiber of \mathfrak{X}/W , the reduction map $NS(X_{\bar{K}}) \rightarrow NS(X)$ is isomorphic and the inclusion $\text{Aut}(X_{\bar{K}}) \hookrightarrow \text{Aut}(X)$ is of finite index. Using this fact, we have that $\text{Im } \rho_X$ and $\text{Im } \chi_{cris, X}$ are finite ([12]). Moreover If n is the order of $\chi_{cris, X}(\alpha)$, $\phi(n)$ is at most the rank of $T_{cris}(X)$.

When X is a K3 surface of arbitrary height over k , an automorphism $\alpha \in \text{Aut}(X)$ is tame if α is of finite order and the order of α is not divisible by the base characteristic p . It is known that if p is greater than 11, any automorphism of finite order of X is tame ([6], Theorem 2.1.). If X is of finite height, we say an automorphism $\alpha \in \text{Aut}(X)$ is weakly tame if the order of $\chi_{cris, X}(\alpha)$ is not divisible by p . A tame automorphism is weakly tame. We say X is weakly tame if the order of $\text{Im } \chi_{cris, X}$ is not divisible by p . Since the rank of $T_{cris}(X)$ is less than 22, if $p \geq 23$, any K3 surface of finite height is weakly tame.

Let X be a K3 surface over k . We say an automorphism $\alpha \in \text{Aut}(X)$ is liftable over W if there is a scheme lifting \mathfrak{X}/W of X/k and a W -automorphism $\mathfrak{a} : \mathfrak{X} \rightarrow \mathfrak{X}$ such

that the restriction of \mathfrak{a} on the special fiber $\mathfrak{a}|_X$ is equal to α . In this paper, we prove that the following theorem.

Theorem 3.3. Let X be a K3 surface over k . If X is of finite height and $\alpha \in \text{Aut}(X)$ is weakly tame, α is liftable over W . If X is supersingular and $\alpha \in \text{Aut}(X)$ is non-symplectic tame, α is liftable over W . If X is supersingular of Artin-invariant at least 2 and $\alpha \in \text{Aut}(X)$ is symplectic tame, α is liftable over W .

Also, for a weakly tame K3 surface, there exists a Neron-Severi group preserving lifting which lifts all the automorphisms.

Theorem 3.7. Let X be a weakly tame K3 surface over k . Then there exists a Neron-Severi group preserving lifting \mathfrak{X}/W of X such that the reduction map $\text{Aut}(\mathfrak{X} \otimes K) \rightarrow \text{Aut}(X)$ is isomorphic.

In a previous work ([12]), we prove that, if k is an algebraic closure of a finite field, X is of finite height and N is the order of $\text{Im } \rho_X$, the rank of $T_{\text{cris}}(X)$ is a multiple of $\phi(N)$. Moreover if X is weakly tame, $p_{\text{cris},X}$ is isomorphic. Using Theorem 3.3, we can prove the same results holds over an arbitrary algebraically closed field.

Corollary 3.5. Let X be a K3 surface of finite height over k . If α is a weakly tame automorphism of X and $\rho_X(\alpha) = id$, then $\chi_X(\alpha) = id$. If X is weakly tame, the projection $p_{\text{cris},X} : \text{Im } \chi_X \rightarrow \text{Im } \rho_X$ is an isomorphism.

Corollary 3.6. Let X be a K3 surface of finite height over k . When N is the order of $\text{Im } \rho_X$, the rank of $T_{\text{cris}}(X)$ is a multiple of $\phi(N)$.

Due to this result, for a weakly tame K3 surface X , $\text{Im } \chi_{\text{cris},X}$ is finite cyclic.

When Σ is a finite set of positive integers $\{13, 17, 19, 25, 27, 32, 33, 40, 44, 50, 66\}$, it is known that, for $N \in \Sigma$, there is a unique complex algebraic K3 surface X_N equipped with a purely non-symplectic automorphism of order N , g_N . A precise elliptic surface model of X_N is known and X_N is defined over \mathbb{Q} . Moreover, if p does not divide $2N$, (X_N, g_N) has a good reduction over an algebraic closure of a prime field \mathbb{F}_p . It is also known that if k is an algebraically closed field of characteristic $p \neq 2, 3$, there is a unique K3 surface over k equipped with an automorphism of order 66 ([15]). Using Theorem 3.3, we prove the uniqueness of a K3 surface equipped with a purely non-symplectic automorphism of order $N \in \Sigma$ when p does not divide $2N$. This unique K3 surface is the reduction of X_N over a finite field.

Theorem 4.3. Assume $N \in \Sigma$ and p does not divide $2N$. Then there exists a unique K3 surface equipped with a purely non-symplectic automorphism of order N . This unique K3 surface has a model over a finite field.

Acknowledgment

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2011-0011428).

2 Deformation of a K3 surface

In this section we review some results on the deformation of K3 surfaces over odd characteristic. For the detail we refer to [24], [4], [5].

Let k be an algebraically closed field of odd characteristic p . Let W be the ring of Witt vectors of k and K be the fraction field of W . Assume X is a K3 surface defined over k . The deformation space of X over artin W -algebras is an affine smooth formal scheme of 20 dimension over W . Let $B = W[[t_1, \dots, t_{20}]]$ and $\mathcal{S} = \text{Spf } B$ be the deformation space of X . Let $\pi : \mathcal{X} \rightarrow \mathcal{S}$ be the universal family over \mathcal{S} . For A , an artin W -algebra whose residue field is isomorphic to k , the set of isomorphic classes of deformation of X over A is $\mathcal{S}(A) = \text{Hom}_{W, \text{cont}}(B, A)$. The second derham cohomology $H = H_{dr}^2(\mathcal{X}/\mathcal{S})$ is a vector bundle of rank 22 on \mathcal{S} . The vector bundle H is equipped with the Hodge filtration

$$H = \text{Fil}^0 \supset \text{Fil}^1 \supset \text{Fil}^2 \supset 0$$

and the Gauss-Manin connection

$$\nabla : H \rightarrow H \otimes_B \Omega_{\mathcal{S}/W}^1.$$

Here Fil^1 and Fil^2 are vector bundles on \mathcal{S} of rank 21 and of rank 1 respectively. The cup product gives a perfect pairing $H \otimes H \rightarrow \mathcal{O}_{\mathcal{S}}$. The graded module of the filtration $gr^i = \text{Fil}^i / \text{Fil}^{i+1}$ is a vector bundle and there is a natural isomorphism

$$gr^i \simeq R^{2-i} \pi_* \Omega_{\mathcal{X}/\mathcal{S}}^i.$$

With respect to the cup product pairing,

$$(\text{Fil}^1)^\perp = \text{Fil}^2 \text{ and } (\text{Fil}^2)^\perp = \text{Fil}^1.$$

By the Griffith transversality, we have

$$\nabla(\text{Fil}^2) \subset \text{Fil}^1 \otimes \Omega_{\mathcal{S}/W}^1.$$

This induces an $\mathcal{O}_{\mathcal{S}}$ -linear morphism

$$gr^2 \nabla : gr^2 \rightarrow gr^1 \otimes \Omega_{\mathcal{S}/W}^1.$$

It is known that $gr^2 \nabla$ is an isomorphism ([4], Proposition 2.4.).

Any $f \in \mathcal{S}(W)$ is corresponding to a formal lifting of X over $\mathrm{Spf} W$, $\mathfrak{X}_f \rightarrow \mathrm{Spf} W$. There is a canonical isomorphism

$$\lambda_f : f^*H = H_{dr}^2(\mathfrak{X}_f/W) \simeq H_{cris}^2(X/W).$$

Through λ_f , the Hodge filtration on $H_{dr}^2(\mathfrak{X}_f/W)$,

$$H_{dr}^2(\mathfrak{X}_f/W) \supset f^*Fil^1 \supset f^*Fil^2$$

gives a filtration on $H_{cris}^2(X/W)$. Let $M_f^i = \lambda_f(f^*Fil^i)$ be a sub module of $H_{cris}^2(X/W)$. A line bundle L on X extends on \mathfrak{X}_f if and only if the crystalline cycle class of L , $c(L) \in H_{cris}^2(X/W)$ is contained in M_f^1 ([24], Proposition 1.12). The rank 1 submodule $M_f^2 \subset H_{cris}^2(X/W)$ satisfies the following conditions.

1. $M_f^2 \otimes k = H^0(X, \Omega_{X/k}^2)$ through the isomorphism $H_{cris}^2(X/W) \otimes k \simeq H_{dr}^2(X/k)$.
2. M_f^2 is isotropic for the cup product pairing.
3. For the canonical Frobenius morphism $\mathbf{F} : H_{cris}^2(X/W) \rightarrow H_{cris}^2(X/W)$, $\mathbf{F}(M_f^2) \subset p^2H_{cris}^2(X/W)$ and $\mathbf{F}(M_f^2) \not\subset p^3H_{cris}^2(X/W)$.

Let us fix a basis v_1, \dots, v_{22} of $H_{cris}^2(X/W)$ satisfying $v_1 \in H_f^2$ and $v_2, \dots, v_{21} \in H_f^1$. Note that $\mathbf{F}(v_1) \in p^2H_{cris}^2(X/W) - p^3H_{cris}^2(X/W)$, $\mathbf{F}(v_i) \in p^2H_{cris}^2(X/W) - p^3H_{cris}^2(X/W)$ for $2 \leq i \leq 21$ and $\mathbf{F}(v_{22}) \notin p^2H_{cris}^2(X/W)$. Since the cup product pairing is perfect and the orthogonal complement of H_f^2 is H_f^1 , we may assume $v_1 \cdot v_{22} = 1$. Assume M is a submodule of $H_{cris}^2(X/W)$ of rank 1 satisfying the condition 1 and the condition 2 above. There exists a unique element

$$v_M = v_1 + \sum_{i=2}^{22} a_i v_i \in M, \quad (a_i \in W).$$

We can easily check that $a_i \in pW$ for $2 \leq i \leq 21$, $a_{22} \in p^2W$ and a_{22} is uniquely determined by a_2, \dots, a_{21} . Since $\mathbf{F}(M_f^1) \subset p^2H_{cris}^2(X/W)$, the condition 3 is automatically satisfied for M . Let \mathcal{M} be the set of rank 1 submodules of $H_{cris}^2(X/W)$ satisfying the condition 1 and the condition 2. The correspondence $M \mapsto (v_2, \dots, v_{21})$ gives a bijection between \mathcal{M} and $(pW)^{20}$, so we may regard $\mathcal{M} = (pW)^{20}$. When g is another element in $\mathcal{S}(W)$ and \mathfrak{X}_g/W is the corresponding lifting, M_g^2 is an element of \mathcal{M} . Let $\Phi : \mathcal{S}(W) \rightarrow \mathcal{M}$ be the function $g \mapsto M_g^2$.

Proposition 2.1 (Local Torelli theorem). The function $\Phi : \mathcal{S}(W) \rightarrow \mathcal{M}$ is bijective.

Proof. Let us fix a morphism $f : B \rightarrow W \in \mathcal{S}(W)$ such that $f(t_i) = 0$ for all i . We choose x , a generator of Fil^2 . Let us denote the differential

$$\nabla(d/dt_i) : H \rightarrow H$$

by D_i . Since $gr^2\nabla$ is isomorphic, we may choose a basis v_1, \dots, v_{22} of $H_{cris}^2(X/W)$ as above such that

$$v_1 = \lambda_f(f^*x) \text{ and } v_i = \lambda_f(f^*D_i x) \text{ for } 2 \leq i \leq 21.$$

Assume $g \in \mathcal{S}(W)$ and $g(t_i) = pa_i \in pW$. The $\mathcal{O}_{\mathcal{S}}$ -module $H = H_{dr}^2(\mathcal{X}/\mathcal{S})$ with the Gauss-Manin connection is an F -cyrstal in sense of [5]. Since $f \otimes k = g \otimes k$, there is an isomorphism

$$\chi(g, f) : H_{dr}^2(\mathfrak{X}_g/W) = g^*H \simeq f^*H = H_{dr}^2(\mathfrak{X}_f/W).$$

Because the Gauss-Manin connection on H is the connection associated to $R^2\pi_{cris,*}\mathcal{O}_{\mathcal{X}}$ ([2], Proposition V 3.6.4), the isomorphism $\chi(f, g)$ makes the following diagram commutes

$$\begin{array}{ccc} H_{dr}^2(\mathfrak{X}_g/W) & \xrightarrow{\chi(g, f)} & H_{dr}^2(\mathfrak{X}_f/W) \\ & \searrow \lambda_g & \downarrow \lambda_f \\ & & H_{cris}^2(X/W). \end{array}$$

Precisely $\chi(g, f)$ is given as follow ([5], Lemme 1.1.2.). When $m = (m_1, \dots, m_{20}) \in \mathbb{N}^{20}$ is a multi index, we denote $D^m = D_1^{m_1} \dots D_{20}^{m_{20}}$. Note that since ∇ is an integrable connection, $D_i D_j = D_j D_i$ for any i, j . Let $\gamma_i : pW \rightarrow W$ be the divided power given by $\gamma_i(a) = a^i/i!$. Then

$$\chi(g, f)(g^*y) = \sum_m \gamma_{m_1}(pa_1) \dots \gamma_{m_{20}}(pa_{20}) f^*(D^m y)$$

for any $y \in H$. The above summation is taken over all the multi index m . We set

$$\lambda_g(g^*x) = \lambda_f(\chi(g, f)(f^*x)) = \sum_i h_i v_i.$$

Here $h_i \in W[[a_1, \dots, a_{20}]]$ is a formal series in a_i . In this case,

$$h_1 = 1 + p^2 k_1$$

and

$$h_i = pa_i + p^2 k_i \quad (2 \leq i \leq 21)$$

where all $k_i (1 \leq i \leq 21)$ are formal series which begins at degree 2 terms. Since $\lambda_g(g^*x)$ is a generator of M_g^2 , $\Phi(g) = h_1^{-1}(h_2, \dots, h_{21}) \in (pW)^{20}$. By the Hensel lemma, the claim follows. \square

3 Lifting of an automorphism

Assume X is a K3 surface over k and α is an automorphism of X . Let \mathfrak{X}_f/W be the formal lifting of X over W associated to $f \in \mathcal{S}(W)$.

Lemma 3.1 (c.f. [24], Corollary 2.5.). An automorphism $\alpha \in \text{Aut}(X)$ extends to \mathfrak{X}_f/W if and only if $\alpha^*|_{H_{cris}^2(X/W)}$ preserves M_f^2 .

Proof. The only if part is trivial. We assume $\alpha^*(M_f^2) = M_f^2$. Let \mathfrak{X}_g/W be the pull back of the lifting \mathfrak{X}_f/W of X/k through the isomorphism α . Then there is a W -isomorphism $\mathfrak{a} : \mathfrak{X}_g \rightarrow \mathfrak{X}_f$ and we have a Cartesian diagram

$$\begin{array}{ccc} X & \hookrightarrow & \mathfrak{X}_g \\ \downarrow \alpha & & \downarrow \mathfrak{a} \\ X & \hookrightarrow & \mathfrak{X}_f. \end{array}$$

Since the isomorphism λ_f and λ_g are functorial, the following diagram commutes.

$$\begin{array}{ccc} H_{dr}^2(\mathfrak{X}_f/W) & \xrightarrow{\mathfrak{a}^*} & H_{dr}^2(\mathfrak{X}_g/W) \\ \downarrow \lambda_f & & \downarrow \lambda_g \\ H_{cris}^2(X/W) & \xrightarrow{\alpha^*} & H_{cris}^2(X/W). \end{array}$$

Because $\mathfrak{a}^*H^0(\mathfrak{X}_f, \Omega_{\mathfrak{X}_f/W}^1) = H^0(\mathfrak{X}_g, \Omega_{\mathfrak{X}_g/W}^1)$ and $\alpha^*M_f^2 = M_f^2$ by the assumption,

$$M_g^2 = \lambda_g(\mathfrak{a}^*H^0(\mathfrak{X}_f, \Omega_{\mathfrak{X}_f/W}^1)) = \alpha^*(\lambda_f(H^0(\mathfrak{X}_f, \Omega_{\mathfrak{X}_f/W}^1))) = M_f^2.$$

By Proposition 2.1, $f = g$ and the automorphism α extends to an automorphism \mathfrak{a} of \mathfrak{X}_f . \square

Remark 3.2. In the above lemma, if \mathfrak{X}_f is algebraizable then α extends to the algebraic model of \mathfrak{X}_f .

Theorem 3.3. Let X be a K3 surface over k . If X is of finite height and $\alpha \in \text{Aut}(X)$ is weakly tame, α is liftable over W . If X is supersingular and $\alpha \in \text{Aut}(X)$ is non-symplectic tame, α is liftable over W . If X is supersingular of Artin-invariant at least 2 and $\alpha \in \text{Aut}(X)$ is symplectic tame, α is liftable over W .

Proof. By Proposition 2.2 and Lemma 3.1, in each case, it is enough to find $M \in \mathcal{M}$ and an ample line bundle V of X such that $\alpha^*M = M$ and M is orthogonal to $c(V) \in H_{cris}^2(X/W)$.

Assume X is of finite height h and α is weakly tame. We fix an F -crystal decomposition

$$H_{cris}^2(X/W) = H_{cris}^2(X/W)_{[1-1/h]} \oplus H_{cris}^2(X/W)_{[1]} \oplus H_{cris}^2(X/W)_{[1+1/h]}$$

and an identification

$$H_{cris}^2(X/W)_{[1-1/h]} = W[F, V]/(FV = p, F = V^{h-1}).$$

Let

$$\pi : H_{cris}^2(X/W) \twoheadrightarrow H_{cris}^2(X/W) \otimes k \simeq H_{dr}^2(X/k)$$

be the canonical projection. We denote the Hodge filtration on $H_{dr}^2(X/k)$ by $F^\cdot H_{dr}^2(X/k)$. Since

$$\mathbf{F}(H_{cris}^2(X/W)_{[1]} \oplus H_{cris}^2(X/W)_{[1+1/h]}) \subset pH_{cris}^2(X/W)$$

and

$$H_{cris}^2(X/W)_{[1-1/h]}/V \simeq H^2(X, \mathcal{O}_X),$$

we have

$$\pi(VH_{cris}^2(X/W)_{[1-1/h]} \oplus H_{cris}^2(X/W)_{[1]} \oplus H_{cris}^2(X/W)_{[1+1/h]}) = F^1 H_{dr}^2(X/k).$$

Let $v \in H_{cris}^2(X/W)_{[1+1/h]}$ be the dual of $1 \in W[F, V]/(FV = p, F = V^{h-1})$ with respect to the base $1, V, \dots, V^{h-1}$ of $H_{cris}^2(X/W)_{[1-1/h]}$. Then

$$\pi(v) \in (F^1 H_{dr}^2(X/k))^\perp = F^2 H_{dr}^2(X/k) \subset H_{dr}^2(X/k)$$

and

$$F^2 H_{dr}^2(X/k) \subset H_{cris}^2(X/W)_{[1+1/h]} \otimes k.$$

Lemma 3.4. Let L be a finite free W -module and $\psi : L \rightarrow L$ be an automorphism of L of finite order coprime to p . Then there is a basis of L consisting of eigenvectors for ψ . If $v \in L \otimes k$ is an eigenvector of $\psi|_{(L \otimes k)}$, there is an eigenvector $\hat{v} \in L$ such that $\hat{v} \otimes k = v$.

Proof. Let N be the order of ψ . Then the polynomial $t^N - 1 \in W[t]$ splits completely. Therefore when L_ζ is the eigenspace of (L, ψ) for an eigenvalue ζ , we have a decomposition

$$L = \bigoplus_{\zeta} L_\zeta.$$

The claim follows easily. \square

Since α is weakly tame and $H_{cris}^2(X/W)_{[1+1/h]}$ is a direct factor of $T_{cris}(X)$, the order of $\alpha^*|_{H_{cris}^2(X/W)_{[1+1/h]}}$ is not divisible by p . Because $F^2 H_{dr}^2(X/k)$ is one dimensional and is invariant for α^* , it follows that, by the above lemma, there is a rank 1 α^* -stable primitive submodule M of $H_{cris}^2(X/W)_{[1+1/h]}$ such that $\pi(M) = F^2 H_{dr}^2(X/k)$. Because $H_{cris}^2(X/W)_{[1+1/h]}$ is isotropic for the cup product, M is an element of \mathcal{M} . Let $f \in \mathcal{S}(W)$ be the lifting of X such that $M_f^2 = M$. Then $M_f^2 \perp H_{cris}^2(X/W)_{[1]}$ and $c(NS(X)) \otimes W$ is a submodule of $H_{cris}^2(X/W)_{[1]}$, so all the line bundles of X extend to \mathfrak{X}_f . In particular, \mathfrak{X}_f is algebraizable and α is liftable over W . Note that \mathfrak{X}_f is a Neron-Severi group preserving lifting of X .

Now assume X is supersingular and α is non-symplectic and tame. Let

$$H_{cris}^2(X/W) = \bigoplus_{\zeta} L_\zeta$$

be the eigenspace decomposition for $\alpha^*|_{H_{cris}^2(X/W)}$. We assume $F^2 H_{dr}^2(X/k) \subset \pi(L_{\zeta_0})$ for some eigenvalue $\zeta_0 \neq 1$. Then $\rho_X(\alpha) = \bar{\zeta}_0$, where $\bar{\zeta}_0$ is the reduction of ζ_0 in k . If

$\zeta_0 \neq -1$, L_{ζ_0} is isotropic and there is a rank 1 primitive submodule $M \subset L_{\zeta_0}$ satisfying $\pi(M) = F^2 H_{dr}^2(X/k)$. Then M is an element of \mathcal{M} . If $\zeta_0 = -1$, $\rho_X(\alpha) = -1$ and by the Serre duality, $\alpha^*|(H_{dr}^2(X/k)/F^1) = -1$. We set $l_{-1} = \pi(L_{-1})$. The pairing on l_{-1} is non-degenerate. The rank of L_{-1} is at least 2 and

$$l_{-1} \not\subset F^1 H_{dr}^2(X/k) = (F^2 H_{dr}^2(X/k))^\perp.$$

Let us choose $0 \neq x \in F^2 H_{dr}^2(X/k)$ and $y \in l_{-1}$ such that $x \cdot y = 1$. Let u and v be liftings of x and y in L_{-1} satisfying $u \cdot v = 1$. Since $u \cdot u$ is divisible by p , by the Hensel lemma, there is $a \in W$ such that $u + pav \in L_{-1}$ is isotropic. If M is a submodule of L_{-1} generated by $u + pav$, M is an element of \mathcal{M} . Let $f \in \mathcal{S}(W)$ be the formal lifting of X over W such that $M = M_f^2$. Because α is of finite order, there is an α^* -stable ample line bundle of X , V . Then $c(V) \in L_1$ and $c(V) \perp L_{\zeta_0} \supset M$. Therefore the formal lifting \mathfrak{X}_f is algebraizable and α is liftable over W .

Assume X is supersingular of Artin-invariant at least 2 and α is symplectic and tame. We set $l_1 = \pi(L_1)$. The pairing on l_1 is non-degenerate. By the assumption, $F^2 H_{dr}^2(X/k) \subset l_1$. Let V be a primitive α^* -ample bundle. Then $c(V) \in L_1$ and $\pi(c(V)) \neq 0$. Let x be a non-zero element of $F^2 H_{dr}^2(X/k)$ and $y = \pi(c(V))$. By [24], Proposition 2.2, x and y are linearly independent. We denote the kernel of $l_1 \rightarrow H^2(X, \mathcal{O}_X)$ by $F^1 l_1$. $F^1 l_1$ is of codimension 1 in l_1 . Note that $x, y \in F^1 l_1$ and $F^1 l_1$ is the orthogonal complement of x in l_1 . Let $c(V)^\perp$ be the orthogonal complement of $c(V)$ in L_1 . Suppose the self intersection of $c(V)$ is not divisible by p . Since $x \cdot y = 0$,

$$F^2 H_{dr}^2(X/k) \subset \pi(c(V)^\perp)$$

and

$$\pi(c(V)^\perp) \not\subset F^1 H_{dr}^2(X/k).$$

Therefore the rank of $c(V)^\perp$ is at least 2 and as above there is a rank 1 submodule M of $c(V)^\perp$ such that $\pi(M) = F^2 H_{dr}^2(X/k)$. The formal lifting corresponding to M is algebraizable and α is liftable to the scheme lifting corresponding to M . Suppose the self intersection of $c(K)$ is divisible by p . Then y is isotropic. Since x and y are linearly independent, there is $y \neq z \in F^1 l_1$ such that $z \cdot y = 1$. Hence the dimension of $F^1 l_1$ is at least 3 and the rank of L_1 is at least 4. Let v and u be arbitrary liftings of x and z in L_1 respectively. Then $v \cdot u$ is divisible by p and $c(K) \cdot u$ is a unit. We choose $w \in L_1$ such that $v \cdot w$ is a unit. We can find $a, b \in W$ satisfying

$$v + au, w + bu \in (c(K))^\perp.$$

Since $v \cdot c(K)$ is divisible by p , $a \in pW$ and $(v + au) \cdot (w + bu)$ is a unit. Then inside $(c(K))^\perp$, we can find a rank 1 isotropic submodule M such that $\pi(M) = F^2 H_{dr}^2(X/k)$. The formal lifting associated to M is algebraizable and α is liftable to the scheme lifting associated to M . \square

Corollary 3.5. Let X be a K3 surface of finite height over k . If α is a weakly tame automorphism of X and $\rho_X(\alpha) = id$, then $\chi_X(\alpha) = id$. If X is weakly tame, the projection $p_{cris, X} : \text{Im } \chi_X \rightarrow \text{Im } \rho_X$ is an isomorphism.

Proof. Since α is weakly tame, as in the proof of the above theorem, there is a Neron-Severi group preserving lifting \mathfrak{X}/W of X equipped with an automorphism $\mathfrak{a} : \mathfrak{X} \rightarrow \mathfrak{X}$ satisfying $\mathfrak{a} \otimes k = \alpha$. Let X_K/K be the generic fiber of \mathfrak{X}/W . By the assumption $\mathfrak{a}^*|H^0(X_K, \Omega_{X_K/K}) = id$. Since K is of characteristic 0, $\mathfrak{a}^*|T(X_K) = id$. But there is a functorial isomorphism

$$H_{dr}^2(X_K/K) \simeq H_{cris}^2(X/W) \otimes K,$$

so $\alpha^*|T_{cris}(X) = id$. The later part follows easily. \square

Corollary 3.6. Let X be a K3 surface of finite height over k . When N is the order of $\text{Im } \rho_X$, the rank of $T_{cris}(X)$ is a multiple of $\phi(N)$.

Proof. Let α be an automorphism of X such that $\rho_X(\alpha)$ generates $\text{Im } \rho_X$. We assume the order of $\chi_{cris,X}(\alpha)$ is $p^r M$ where M is a positive integer which is not divisible by p . Then α^{p^r} is weakly tame and M is equal to N by the above corollary. Replacing α by α^{p^r} , we may assume α is weakly tame. Then there exist a Neron-Severi group preserving lifting \mathfrak{X}/W and a lifting of α , $\mathfrak{a} : \mathfrak{X} \rightarrow \mathfrak{X}$. Since the order of $\rho_{X_K}(\mathfrak{a})$ is N , the rank of $T(X_K)$ is a multiple of $\phi(N)$. The rank of $T(X_K)$ is equal to the rank of $T_{cris}(X)$ and the claim follows. \square

Theorem 3.7. Let X be a weakly tame K3 surface over k . There exists a Neron-Severi group preserving lifting \mathfrak{X}/W of X such that the reduction map $\text{Aut}(\mathfrak{X} \otimes K) \rightarrow \text{Aut}(X)$ is isomorphic.

Proof. Let h be the height of X . Let α be a weakly tame automorphism of X such that $\rho_X(\alpha)$ generates $\text{Im } \rho_X$. Then by Corollary 3.5, $\chi_{cris,X}(\alpha)$ generates $\text{Im } \chi_{cris,X}$. As in the proof of Theorem 3.3, we can find $M \in \mathcal{M}$ inside $H_{cris}^2(X/W)_{[1+1/h]}$ which is α^* -stable. Let \mathfrak{X}/W be the lifting of X corresponding to M . Then α is liftable to \mathfrak{X} . For any $\beta \in \text{Aut}(X)$, $\chi_{cris,X}(\beta) = \chi_{cris,X}(\alpha^i)$ for some integer i . Since

$$H_{cris}^2(X/W)_{[1+1/h]} \subset T_{cris}(X),$$

M is stable for β^* and β is liftable to \mathfrak{X} . Therefore

$$\text{Aut}(\mathfrak{X} \otimes K) = \text{Aut}(\mathfrak{X}) \rightarrow \text{Aut}(X)$$

is surjective. \square

Remark 3.8. Assume p is at least 5 and X is a supersingular K3 surface of Artin-invariant 1 over k . Then $\text{Im } \rho_X$ is a cyclic group of order $p+1$ ([13]). Hence if $p > 60$, $\phi(p+1) > 21$ and there is an automorphism of X which can not be lifted over characteristic 0. It is also known that for a supersingular K3 surface of Artin-invariant 1 over a field of characteristic 3, there is an automorphism which can not be lifted over characteristic 0 ([7]). We can ask whether for any supersingular K3 surface, there is an automorphism which can not be lifted over characteristic 0.

4 Non-symplectic automorphisms

Let k be an algebraically closed field of odd characteristic p whose cardinality is equal to or less than the cardinality of the real numbers. Let W be the ring of Witt-vectors of k and K be the fraction field of W . Let \bar{K} be an algebraic closure of K . We fix an isomorphism $\bar{K} \simeq \mathbb{C}$. Let $\Sigma = \{13, 17, 19, 25, 27, 32, 33, 40, 44, 50, 66\}$ be a finite set of positive integers. The following is known.

Theorem 4.1 ([16], [18], [23], [26]). If $N \in \Sigma$, there exists a unique complex algebraic K3 surface X_N equipped with a purely non-symplectic automorphism of order N , g_N up to isomorphism. X_N has a model over \mathbb{Q} and if a prime number p does not divide $2N$, (X_N, g_N) has a good reduction $(X_{N,p}, g_{N,p})$ over an algebraic closure of a prime field of characteristic p .

In the case of $N = 66$, the following result over positive characteristic is also known.

Theorem 4.2 ([15]). If the characteristic of k is not 2 or 3, there is a unique K3 surface equipped with an automorphism of order 66.

Note that the above result covers a wild case of characteristic 11.

Using Theorem 3.3, we prove the uniqueness of a K3 surface over k equipped with a purely non-symplectic tame automorphism of order N for $N \in \Sigma$.

Theorem 4.3. Assume $N \in \Sigma$ and p does not divide $2N$. Then there exists a unique K3 surface equipped with a purely non-symplectic automorphism of order N up to isomorphism. This unique K3 surface has a model over a finite field.

Proof. The existence is guaranteed by Theorem 4.1. Now assume X is a K3 surface over k and $\alpha \in \text{Aut}(X)$ is purely non-symplectic of order N . Since α is non-symplectic tame, by Theorem 3.3, there exists a scheme lifting \mathfrak{X}/W of X and an automorphism $\mathfrak{a} : \mathfrak{X} \rightarrow \mathfrak{X}$ such that $\mathfrak{a} \otimes k = \alpha$. Then $\mathfrak{X} \otimes \mathbb{C}$ is a complex K3 surface equipped with a purely non-symplectic automorphism of order N , so $\mathfrak{X} \otimes \mathbb{C} \simeq X_N$. It follows that X is isomorphic to $X_{N,p} \otimes k$. \square

References

- [1] Artebani, M., Sarti, A. and Taki, S. K3 surfaces with non-symplectic automorphisms of prime order, *Math.Z.* 268, 2011, 507–533.
- [2] Berthelot, P. Cohomologie cristalline des schémas de caractéristique $p > 0$, *Lecture Notes in Math.* 407, 1974.
- [3] Francois, C. The Tate conjecture for K3 surfaces over finite fields, *Invent. Math.* 194, 2013, 119–145.
- [4] Deligne, P. Relèvement des surfaces K3 en caractéristique nulle, *Lecture Notes in Math.* 868, *Algebraic Surfaces*, 58–79, 1981.

- [5] Deligne, P. Cristaux ordinaires et coordonnées canoniques, *Lecture Notes in Math.* 868, Algebraic Surfaces, 80–137, 1981.
- [6] Dolgachev, I. and Keum, J. Finite group of symplectic automorphisms of K3 surfaces in positive characteristic, *Ann. of Math.* 169, 269–313, 2009.
- [7] Esnault, H. and Oguiso, K. Non-liftability of automorphism groups of a K3 surface in positive characteristic, *arXiv:1406.2761*.
- [8] Illusie, L. Report on crystalline cohomology, *Algebraic Geometry, Proc. Sympos. Pure Math* 29, 1975, 459–478.
- [9] Illusie, L. Complexe de de Rham-Witt et cohomologie cristalline, *Ann. ENS 4serie* 12, 1979, 501–661.
- [10] Illusie, L. and Raynaud, M. Les suites spectrales associees au complexe de de Rham-Witt, *Pub. IHES* 57, 1983, 73–212
- [11] Jang, J. Neron-Severi group preserving lifting of K3 surfaces and applications, *arXiv:1306.1596*
- [12] Jang, J. The representation of the automorphism groups on the transcendental cycles and the Frobenius invariants of K3 surfaces, *arxiv* 1312.7634.
- [13] Jang, J. Representations of the automorphism group of a supersingular K3 surface of Artin-invariant 1 over odd characteristic, *J. of Chungcheong Math. Soc.* 27, No.2, 287–295, 2014.
- [14] Katz, N. Slope filtration of F-crystal, *Astérisque* 63, 1979, 113–163.
- [15] Keum, J. K3 surfaces with an automorphism of order 66, the maximum possible, *arXiv:1302.6803*.
- [16] Kondo, S. Automorphisms of algebraic K3 surfaces which acts trivially on Picard groups, *J. Math. Soc. Japan* 44, No. 1, 75–98, 1992.
- [17] Lieblich, M. and Maulik, D. A note on the cone conjecture for K3 surfaces in positive characteristic, *arXiv:1102.3377*.
- [18] Machida, N. and Oguiso, K. On K3 surfaces admitting finite non-symplectic group actions, *J. Math. Sci. Univ. Tokyo* 5, 1998, 273–297.
- [19] Madapusi Pera, K. The Tate conjecture for K3 surfaces in odd characteristic, *arXiv:1301.6326*.
- [20] Maulik, D. Supersingular K3 surfaces for large primes, *arXiv:1203.2889*.
- [21] Nukulin, V. V. Finite group of automorphisms of Kählerian K3 surfaces, *Trudy Moskov. Mat. Obshch* 38, 1979, 75–137.

- [22] Nygaard,N.O. and Ogus,A. Tate conjecture for K3 surfaces of finite height, Ann. of Math.(2) 122, 1985, 461–507.
- [23] Ogus,K and Zhang,D. On Vorontsov’s theorem on K3 surfaces with non-symplectic group actions, Proc. A.M.S. 128, 2000, 1571–1580.
- [24] Ogus,A. Supersingular K3 crystal, Astérisque 64, 1979, 3–86
- [25] Rudakov,A.N. and Shafarevich,I.R. Surfaces of type K3 over fields of finite characteristic, Current problems in mathematics, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii vol. 18, 1981, 115–207.
- [26] Taki,S. On Ogus’s K3 surface, Journal of pure and applied algebra 218, 2014, 391–394.

J.Jang
 Department of Mathematics
 University of Ulsan
 Daehakro 93, Namgu Ulsan 680-749, Korea

 jmjang@ulsan.ac.kr